Pro-étale cohomology of Drinfeld's symmetric space

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Outline

1. Logarithm exact sequence
2. Steinberg representations
3. Some results from Colmez–Dospinescu–Niziol
4. Computing $H^i_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$
5. Computing invariant module

Goal of this talk:

$$H^0(\Pi_n, \mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)^*) \cong H^0(\Pi_n, H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})).$$

We will focus on $H^i(\Pi_n, H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$ for $i = 1$. The main reference [1] contains results of greater generality.

Recollection

1. $\mathcal{H}_C^{n-1} = \mathbb{P}_C^{n-1} \setminus \mathbb{Q}_p$-rational hyperplanes for C the completion of $\overline{\mathbb{Q}_p}$.
2. $\Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}), \Pi_n = \Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Z}_p)$
3. Fundamental exact sequence:

$$0 \to H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \xrightarrow{\det^*} H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \xrightarrow{b} H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}$$

for $(n, p) \neq (2, 2)$

$$0 \to H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \oplus \mathbb{Z}/2 \xrightarrow{\det^* \oplus \hat{\alpha}} H^1_{\mathrm{cts}}(\mathbb{G}_2, A_2^{**}) \xrightarrow{b} H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}$$

for $(n, p) = (2, 2)$.

Notation: the Tate twist

$\mathbb{Q}_p(1)$: the 1-th Tate twist of $\mathbb{Q}_p$. Topologically, this is $\mathbb{Q}_p$, but the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ factors through $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$.
More explicitly, we have the cyclotomic character map

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$$

by restriction, and the right hand side act on the vector space $\mathbb{Q}_p$. Thus, we obtain a 1-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over $\mathbb{Q}_p$, which we denote as the Tate twist $\mathbb{Q}_p(1)$.
We can define $\mathbb{Z}_p(1)$ using a short exact sequence

$$0 \to \mathbb{Z}_p(1) \to \mathbb{Q}_p(1) \to \mu_{p^\infty} \to 0. \tag{1}$$

Logarithm exact sequence

Lemma (Logarithm exact sequence, [1] 4.2.1)

Let X be a rigid-analytic variety. There is an exact sequence of sheaves of condensed abelian groups on the (pro-)étale site of X:

$$0 \to \mu_{p^\infty} \to \mathcal{O}^{**} \xrightarrow{\log} \mathcal{O} \to 0 \tag{2}$$

where μ_{p^∞} is the sheaf of p-th roots of unity.

We want to commpute $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$. The logarithm exact sequence allows us to approach this via computing $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mu_{p^\infty})$. The short exact sequence

$$0 \to \mathbb{Z}_p(1) \to \mathbb{Q}_p(1) \to \mu_{p^\infty} \to 0$$

further reduces our task to computing $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Q}_p(1))$.

Steinberg representation I

For a profinite set $S = \lim S_i$ and a ring A, let $\mathrm{LC}(S, A)$ be the locally constant function on S with values in A. If there is a topology on A, then we give $LC(S, A) = \lim LC(S_i, A)$ the colimit topology.

Definition (Steinberg representation)

We define

$$\mathrm{St}_1(A) = \frac{\mathrm{LC}(\mathbb{P}^{n-1}(\mathbb{Q}_p), A)}{A}.$$

There is a continuous action of $\mathrm{GL}_n(\mathbb{Q}_p)$ on $\mathrm{St}_1(A)$.

Definitions and results for St_r could be found in the main paper.

Steinberg representation II

Let $\mathrm{St}_1(A)^*$ denote the continuous A-module homomorphisms

$$\mathrm{St}_1(A) \to A.$$

For $l, l' \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$, we view $\delta_l - \delta_{l'}$ as an element of $\mathrm{St}_1(A)^*$, where δ denotes the evaluation map on l.

Definition ($\mathrm{St}_1(\mathbb{Q}_p\,/\,\mathbb{Z}_p)^*$)

We define the $\mathrm{GL}_n(\mathbb{Q}_p)$-module $\mathrm{St}_r(\mathbb{Q}_p\,/\,\mathbb{Z}_p)^*$ using the exact sequence

$$0 \to \mathrm{St}_1(\mathbb{Z}_p)^* \to \mathrm{St}_1(\mathbb{Q}_p)^* \to \mathrm{St}_1(\mathbb{Q}_p\,/\,\mathbb{Z}_p)^* \to 0. \tag{3}$$

Note that this definition is ad-hoc as $\mathbb{Q}_p\,/\,\mathbb{Z}_p$ is not a ring.

Steinberg representation III

Lemma ([1] 5.1.2)

For $n \geq 2$, the subset of $\mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)^*$ fixed by $\mathrm{GL}_n(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is a free $\mathbb{Z}_p$-module of rank 1.

Taking the $\mathrm{GL}_n(\mathbb{Z}_p)$-fixed points, we obtain an injective boundary map associated with the short exact sequence 3:

$$\partial : H^0(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)^*) \to H^1(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Z}_p)^*).$$

Lemma ([1] 5.1.4)

Let μ be the generator of $H^0(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)^*)$. Then $\partial(\mu) \in H^1(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Z}_p)^*)$ is represented by the cocycle

$$g \mapsto \delta_I - \delta_{g(I)}$$

for $g \in \mathrm{GL}_n(\mathbb{Z}_p)$.

CDN: Étale cohomology I

Definition

We define

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) = \lim H^i_{\text{ét}}(X, \underline{\mathbb{Z}/p^n\mathbb{Z}})$$

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) = H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

Note that this is an ad-hoc definition souped up from the constant sheaf $\underline{\mathbb{Z}/p^n\mathbb{Z}}$ on $X_{\text{ét}}$. We do have a constant sheaf $\mathbb{Q}_p$ on $X_{\text{proét}}$, but

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \to H^i_{\text{proét}}(X, \mathbb{Q}_p)$$

is not always an isomorphism.

CDN: Étale cohomology II

Consider the two short exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_{p^{n+1}} & \longrightarrow & \mathcal{O}^* & \xrightarrow{(-)^{p^{n+1}}} & \mathcal{O}^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (-)^p} & & \downarrow & & \\
0 & \longrightarrow & \mu_{p^n} & \longrightarrow & \mathcal{O}^* & \xrightarrow{(-)^{p^n}} & \mathcal{O}^* & \longrightarrow & 0
\end{array}
$$

Taking limits along the vertical maps, we get another short exact sequence, we get

$$0 \to \mathbb{Z}_p(1) \to \lim_{\leftarrow (-)^p} \mathcal{O}^* \to \mathcal{O}^* \to 0. \tag{4}$$

Definition (Kummer map κ)

We define

$$\kappa : H^0_{et}(\mathcal{H}^{n-1}_C, \mathcal{O}^*) \to H^1_{et}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1))$$

be the connecting homomorphism.

CDN: Étale cohomology III

Theorem (Colmez–Dospinescu–Niziol)

There is a $\Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$*-equivariant isomorphism*

$$r_1 : \mathrm{St}_1(\mathbb{Z}_p)^* \to H^1_{\text{ét}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1)).$$

This is given by

$$r_1(\delta_{l_1} - \delta_{l_2}) = \kappa(l_1/l_2).$$

CDN: Pro-étale cohomology I

For a rigid space X over C, by taking the inverse limit along $\times p$ map, the short exact sequence on $X_{\text{proét}}$ of sheaves

$$0 \to \mu_{p^\infty} \to \mathcal{O}^{**} \xrightarrow{\log} \mathcal{O} \to 0$$

gives another short exact sequence

$$0 \to \mathbb{Q}_p(1) \to \varprojlim_{\times p} \mathcal{O}^{**} \xrightarrow{\log'} \mathcal{O} \to 0.$$

There is a boundary map

$$\partial : \mathcal{O}[-1] \to \mathbb{Q}_p(1).$$

We have a spectral sequence

$$H^1_{\text{ét}}(X, \Omega^j(-j)) \Rightarrow H^{1+j}_{\text{proét}}(X, \mathcal{O})$$

where Ω^1 denotes the sheaf of differential 1-forms. In our case $X = \mathcal{H}^{n-1}_C$, the coherent sheaves of differential j-forms are acyclic, so we get

$$H^1_{\text{proét}}(X, \mathcal{O}) \cong \Omega^1(X)(-1).$$

CDN: Pro-étale cohomology II

Definition (exp map)

We define
$$\exp : \Omega^0(X)(0) \to H^1_{\text{proét}}(X, \mathbb{Q}_p(1)).$$

Theorem (Colmez–Dospinescu–Niziol)

There is a $\Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$-equivariant exact sequence of $\mathbb{Q}_p$-vector space

$$0 \to \frac{\Omega^0(\mathcal{H}_C^{n-1})}{\ker d} \xrightarrow{\exp} H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Q}_p(1)) \to \mathrm{St}_1(\mathbb{Q}_p)^* \to 0$$

This theorem exhibits the pro-étale cohomology of $\mathcal{H}_C^{n-1}$ as an extension of a space of differential forms by the dual of a Steinberg representation. Our goal is to compute $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$, so we need to compute $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mu_{p^\infty})$ in order to use the logarithm exact sequence.

CDN: Pro-étale cohomology III

Corollary ([1] 5.2.5)

There is a $\Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$*-equivariant exact sequence*

$$0 \to \frac{\Omega^0(\mathcal{H}_C^{n-1})}{\ker d} \xrightarrow{\exp'} H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mu_{p^\infty}) \to \mathrm{St}_1(\mathbb{Q}_p \,/\, \mathbb{Z}_p)^* \to 0$$

Proof.

We have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1)) & \xrightarrow{\cong} & \mathrm{St}_1(\mathbb{Z}_p)^* & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow \frac{\Omega^0(\mathcal{H}_C^{n-1})}{\ker d} & \longrightarrow & H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Q}_p(1)) & \longrightarrow & \mathrm{St}_1(\mathbb{Q}_p)^* & \longrightarrow & 0 \end{array}$$

Snake lemma finishes the proof. □

Calculation of $\mathcal{H}_C^{n-1}$ I

We are ready to use the logarithm exact sequence and the previous corollary to deduce our main theorem.

Theorem (5.3.1)

We have a short exact sequence

$$0 \to \mathrm{St}_1(\mathbb{Q}_p \,/\, \mathbb{Z}_p)^*(0) \to H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**}) \to \Omega^{1,cl}(\mathcal{H}_C^{n-1})(-1) \to 0$$

where $\Omega^{1,cl}$ denotes the sheaf of closed differential 1-forms on $\mathcal{H}_C^{n-1}$.

Proof.

Let $\partial^1 : H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}) \to H^2_{\text{proét}}(\mathcal{H}_C^{n-1}, \mu_{p^\infty})$ be the boundary map of the logarithm exact sequence. From the long exact sequence associated to the logarithm sequence on cohomology, there is a short exact sequence

$$0 \to \mathrm{coker}(\partial^0) \to H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**}) \to \ker(\partial^1) \to 0.$$

Proof.

Note that since as p acts on $\mathcal{O}$ invertibly, the composition $\mathcal{O} \to \mu_{p^\infty}[1] \to \mathbb{Z}_p(1)[2]$ is trivial, and we have a dashed lift

$$\begin{array}{ccccc} & & \mathcal{O} & & \\ & \swarrow{\scriptstyle --} & \downarrow{\scriptstyle \partial} & & \\ \mathbb{Q}_p(1)[1] & \longrightarrow & \mu_{p^\infty}[1] & \longrightarrow & \mathbb{Z}_p(1)[2] \end{array}.$$

Therefore, on cohomology level we have

$$H^0_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}) \to H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mu_{p^\infty}),$$

which coincides with

$$\exp : \Omega^0(\mathcal{H}_C^{n-1})(0) \cong H^0_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}) \to H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Q}_p(1)).$$

Calculation of $\mathcal{H}_C^{n-1}$ III

Proof.

We now obtain a diagram of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega^1(\mathcal{H}_C^{n-1})(-1) & \xrightarrow{\cong} & H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \partial^1 & & \\
0 & \longrightarrow & \frac{\Omega^1(\mathcal{H}_C^{n-1})}{\ker d}(-1) & \xrightarrow{\exp'} & H^2_{\text{proét}}(\mathcal{H}_C^{n-1}, \mu_{p^\infty}) & \longrightarrow & \mathrm{St}_2(\mathbb{Q}_p/\mathbb{Z}_p)^*(-1) \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}$$

Snake lemma gives

$$\ker(\partial^1) = \Omega^{1,cl}(\mathcal{H}_C^{n-1})(-1), \ \mathrm{coker}(\partial^0) = \mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*(0).$$

□

Invariant module I

Corollary (5.3.2)

The short exact sequence of pro-étale shaves on $\mathcal{H}_C^{n-1}$:

$$0 \to \mathbb{Z}_p(1) \to \lim_{\leftarrow (-)\times p} \mathcal{O}^{**} \to \mathcal{O}^{**} \to 0$$

induces short exact sequence of Π_n-modules for $m \geq 0$:

$$0 \to H^m_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1)) \to H^m_{\text{proét}}(\mathcal{H}_C^{n-1}, \lim_{\leftarrow (-)\times p} \mathcal{O}^{**}) \to H^m_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$$

In other words, the boundary maps associated with the short exact sequence are 0.

Invariant module II

Proof.

We need to show that

$$H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \lim_{\leftarrow (-)\times p} \mathcal{O}^{**}) \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$$

is a surjective map. First, note that the map factor as

$$H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \lim_{\leftarrow (-)\times p} \mathcal{O}^{**}) \to \lim_{\leftarrow (-)\times p} H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**}) \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$$

Also we note that the first map is surjective by the Milnor sequence. Theorem 9 exhibits $H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$ as a p-divisible group, so the map from the inverse limit to $H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$ is surjective. □

Invariant module III

Corollary (5.3.3)

There is a canonical isomorphism

$$H^0(\Pi_n, \mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)^*) \cong H^0(\Pi_n, H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})).$$

Therefore, for $n \geq 2$, we identify $H^0(\Pi_n, H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**}))$ as a free $\mathbb{Z}_p$-module of rank 1. For $n = 1$, we have $H^0(\Pi_n, H^0_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})) = 0$.

Fix points IV

Proof.

Recall from Theorem 11, we have a short exact sequence

$$0 \to \mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)^*(0) \to H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**}) \to \Omega^{1,cl}(\mathcal{H}_C^{n-1})(-1) \to 0$$

which Π_n acts on. The action of Π_n on $\mathrm{St}_1(\mathbb{Q}_p / \mathbb{Z}_p)$ is through $\mathrm{GL}_n(\mathbb{Z}_p)$. First observe the base change

$$\Omega^{1,cl}(\mathcal{H}_C^{n-1})(-1) = \Omega^{1,cl}(\mathcal{H}^{n-1}) \otimes_{\mathbb{Q}_p} C(-1)$$

where Π_n acts on $C(-1)$, so we get

$$H^0(\Gamma_{\mathbb{Q}_p}, \Omega^{1,cl}(\mathcal{H}_C^{n-1})(-1)) = \Omega^{1,cl}(\mathcal{H}^{n-1}) \otimes H^0(\Gamma_{\mathbb{Q}_p}, C(-1)) = 0.$$

The second isomorphism follows from Theorem 4.4.3 of [2]. Therefore, we obtain the desired isomorphism from the short exact sequence in Theorem 11. □

References

Tobias Barthel, Tomer Schlank, Nathanial Stapleton, Jared Weinstein. *On Hopkins' Picard Group*, 2024. https://arxiv.org/pdf/2407.20958

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